

On Unit Distances in a Convex Polygon

Amol Aggarwal
Saratoga High School
Saratoga, California
July 31, 2010

Abstract

For any convex quadrilateral, the sum of the lengths of the diagonals is greater than the corresponding sum of a pair of opposite sides, and all four of its interior angles cannot be simultaneously acute. In this article, we use these two properties to estimate the number of unit distance edges in convex n -gons and we: (i) exhibit three large groups of cycles formed by unit distance edges that are forbidden in convex n -gons, (ii) prove that the maximum number of unit distances is at most $n \log_2 n + 4n$, thereby improving the best known result by a factor of 2π , and (iii) we show that if we only use these two properties then we will not be able to further improve this bound by more than a factor of four.

1 Introduction

In 1959, Erdős and Moser asked the following question: what is the maximum number of unit distances that can be formed by vertices of a convex n -gon [8]? They conjectured a linear bound, gave a lower bound of $\lfloor 5(n-1)/3 \rfloor$, and gave an upper bound of $O(n^{3/2})$. In [5], Edelsbrunner and Hajnal improved this lower bound to $2n-7$, which later led to Erdős and Fishburn's conjecture in [7] that the number of unit distances in any convex n -gon is less than $2n$. In [15], Szemerédi and Trotter improved the lower bound to $O(n^{4/3})$ and later, Füredi improved this upper bound to $2\pi n \log n$ in [9]. In [4], Brass and Pach prove the upper bound $9.65n \log n$ using induction and geometric constraints different from those provided by Füredi. In [1], Ábrego and Fernández-Merchant proved that the maximum number of unit distances in a centrally symmetric convex n -gon is at most $2n-3$. Finally, in [17], Toussaint connected music and problems in combinatorial geometry by showing how Erdős's problem in [6] that there are at least $\lfloor n/2 \rfloor$ distinct distances in a convex n -gon is related to *deep rhythms*.

Before we discuss the results of the paper, we define the following terms. The following is due to Brass and Pach in [4]. Consider a convex polygon $\mathcal{P} = v_1 v_2 v_3 \cdots v_n$, and let v_1 and v_k be vertices such that there exist two parallel lines passing through v_1 and v_k respectively such that \mathcal{P} lies within this parallel strip, so that v_1 and v_k are *antipodal vertices*. Let $v_{n-i} = u_i$ for all $1 \leq i \leq n-k-1$ and let $n-k-1 = l$. We call the chains $v_1 v_2 v_3 \cdots v_k$ and $u_1 u_2 u_3 \cdots u_l$ *convex chains* of lengths k and l respectively. Create a $k \times l$ matrix, called the *distance matrix*, in which the entry in the i th row and j th column is $d(v_i, u_j)$, where $d(v, u)$ is the Euclidean distance between points u and v in the plane. Define the $0-1$ *matrix* to be the $k \times l$ matrix in which the entry in the i th row and j th column is a 1 if $d(v_i, u_j) = 1$ and a 0 otherwise. We call $v_i u_j$ a *unit edge* if and only if $d(v_i, u_j) = 1$, and we call a *cycle* a

cycle of unit edges. In particular, note that any unit edge must cross the *diameter* v_1v_k since one endpoint is from one convex chain and the other vertex is from the other convex chain. Unit-distanced edges that do not necessarily cross the diameter will be called *unit-distanced edges* or *edges of unit distance*.

A matrix $M = \{m_{ij}\}$ with positive entry values that does not have entries $m_{x_1y_1}$, $m_{x_1y_2}$, $m_{x_2y_1}$, and $m_{x_2y_2}$ so that $x_1 < x_2$ and $y_1 < y_2$ and $m_{x_1y_1} + m_{x_2y_2} \geq m_{x_1y_2} + m_{x_2y_1}$ is said to have the *diagonal property*. This submatrix was also explored in [14] by Pach and Tardos while trying to understand properties of unit-distanced edges formed by n points in the plane. Similarly, if M does not have entries $m_{m_1n_1} \geq m_{m_1n_2}, m_{m_2n_1}$ and $m_{p_2q_2} \geq m_{p_2q_1}, m_{p_1,q_2}$ so that $m_1 \leq m_2, p_1 \leq p_2, n_1 \leq q_1, n_2 \leq q_2, m_1 < p_2$, and $n_1 < q_2$ (the entries, when combined, form an *acute angle submatrix*) is said to have the *obtuse angle property*. Submatrices in which each of these six entries is equal to one were examined by Brass, Karolyi, and Valtr [3]. For instance, the first matrix below is an acute angle submatrix when $a \geq b, c$ and $f \geq d, e$ and the second is an acute angle submatrix when $a \geq b, c$ and $e \geq d, b$.

$$\begin{bmatrix} a & b & & \\ c & & d & \\ & e & f & \end{bmatrix}; \quad \begin{bmatrix} a & & b \\ c & & \\ & d & e \end{bmatrix}$$

In this article, we provide the following results:

Theorem 1: Any convex n -gon can have at most $n \log_2 n + 4n$ edges of unit distance.

Theorem 2: Consider a 0 – 1 matrix in which the 1 entries, when viewed as vertices, form a (non self intersecting) rectilinear polygon. For instance, if

$$A = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ 1 & & 1 & \end{bmatrix}; \quad B = \begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{bmatrix}$$

then the 1s of A , when connected by horizontal and vertical edges, form a rectilinear polygon that looks like a staircase. However, whenever connect the 1s of B with edges, the corresponding figure is self-intersecting, so it is not a polygon. Matrices like A are called *non self intersecting polygon matrices*. Theorem 2 states that there is no non self intersecting polygon matrix that can correspond to two convex chains.

Theorem 3: Any cycle that has a unit edge that does not intersect any other unit edge in this cycle is forbidden.

Theorem 4: There exists a $2^m \times 2^m$ matrix (henceforth called as the *distance-like matrix*) that has positive entries and satisfies the diagonal property and the obtuse angle property with $2^{m-1}(m+1)$ ones.

The proofs of Theorems 1, 2, 3, and 4 are provided in Sections 3, 4, 5, and 6 respectively. Note that Theorem 4 shows that our techniques are not enough to prove the linear bound conjectured by Erdős and Moser. This matrix may correspond to a convex 2^{m+1} -gon partitioned into two convex chains of lengths 2^m each, but we have been unable to prove or disprove this statement. In Section 7, we provide some general discussion on previous results, the results of this paper, and open problems.

2 Preliminary Observations

We first define the $(1, +, -)$ *matrix*. In the i th row and j th column of this matrix, put a 1 if $d(v_i, u_j) = 1$, a + if $d(v_i, u_j) > 1$, and a - if $d(v_i, u_j) < 1$. The same matrix has been mentioned by Pach and Tardos in [14] in order to exhibit properties of unit distances between points in the n plane. Proposition 1 below is due to Brass and Pach [4]. Propositions 2 and 3 have also been used before. In [2], Altman used Proposition 2 to provide the lower bound of $\lfloor n/2 \rfloor$ distinct distances in a convex polygon, whereas Füredi used a variant of Proposition 3 to show that there are at most $2\pi n \log_2 n$ unit-distanced edges in a convex n -gon in [10]. Proposition 2 has also been used in [14] and Proposition 3 in [3].

Proposition 1 (Brass and Pach; [4]): Partition a polygon \mathcal{P} into two convex chains as in Section 1, namely $\mathcal{V} = v_1 v_2 v_3 \cdots v_k$ and $\mathcal{U} = u_1 u_2 u_3 \cdots u_l$. Among the set of edges connecting two vertices from \mathcal{V} or two from \mathcal{U} , at most $2n$ have unit distance.

Proof: Consider parallel lines l and l' through v_1 and v_k such that \mathcal{P} lies completely between l and l' and such that no edge connecting two vertices of \mathcal{P} is perpendicular to l . Let l be parallel to the x -axis in the Cartesian coordinate plane. Now color any unit-distanced edge connecting two vertices from \mathcal{V} or two vertices from \mathcal{U} white if it has positive slope and black if it has negative slope. Moreover, *assign* any unit-distanced edge to its leftmost vertex. It is easily seen that each vertex is the assignment of at most one white edge, and similarly, each vertex is the assignment of at most one black edge, meaning that there are at most $2n$ colored edges, which proves the proposition. ■

Proposition 2: $a_{ij} + a_{kl} < a_{il} + a_{kj}$ when $i < k$ and $j < l$ for any distance matrix $A = \{a_{ij}\}$ that can correspond to two convex chains.

Proof: The inequality in this proposition is equivalent to the inequality $d(i, j) + d(k, l) < d(i, l) + d(k, j)$ when $i < k$ and $j < l$, which follows from the fact that the sum of the lengths of the diagonals of convex quadrilateral $v_i u_j u_l v_k$ is greater than the corresponding sum of any pair of opposite sides. This proposition implies that any distance matrix corresponding to two convex chains must satisfy the distance property. ■

Proposition 3: $a \geq b, c$ and $f \geq d, e$, then it is not possible for a distance matrix corresponding to two convex chains to have the submatrix $\begin{bmatrix} a & b \\ c & \\ & d \\ & e & f \end{bmatrix}$. Similarly, the other acute angle matrices cannot exist in a distance matrix corresponding to a pair of convex chains.

Proof: Suppose that it is possible to have two convex quadrilaterals $v_1v_2v_3v_4$ and $u_1u_2u_3u_4$ so that $d(v_1, u_1) \geq d(v_1, u_2), d(v_2, u_1)$ and $d(v_4, u_4) \geq d(v_3, u_4), d(v_4, u_3)$. Then, $\angle v_2v_1u_1$ is acute since $d(v_2, u_1) \leq d(v_1, u_1)$, and hence $\angle v_4v_1u_1 < \angle v_2v_1u_1$ is also acute. Similarly, every other interior angle of convex quadrilateral $v_1u_1u_4v_4$ is acute, which is impossible. Therefore, such submatrix cannot be contained in a distance matrix corresponding to two convex chains. Therefore, any distance matrix that corresponds to two convex chains must satisfy the obtuse angle property. ■

Corollary 1: Notice that Proposition 2 is a corollary of Proposition 3 when viewed as a $(1, +, -)$ matrix. For instance, Proposition 3 forbids the matrices given below, and Proposition 3 forbids them as well:

$$\begin{bmatrix} + & 1 \\ - & + \end{bmatrix}; \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

3 An improved upper bound

First, we prove the following two lemmas, the first given by Keszegh in [11] and [12], and the second given by Tardos in [16].

Lemma 1 (Keszegh; [11] and [12]): For natural numbers a and b and a $0-1$ matrix M , let $ex(a, b, M)$ be the maximum number of 1 entries in an $a \times b$ $0-1$ matrix that does not have M as a submatrix. Now suppose that A and B are $0-1$ matrices such that the bottom right entry of A is a 1 and the top left entry of B is a 1. Let C be the $0-1$ matrix that contains A in the top left and B in the top right such that the only entry shared between A and B is the 1 in A 's bottom right corner. For instance, if $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix}, \text{ then } C = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & 1 & \end{bmatrix}. \text{ Then, } ex(a, b, A) + ex(a, b, B) \geq ex(a, b, C).$$

Proof: Suppose that the $a \times b$ matrix contains at least $ex(a, b, A) + 1 + ex(a, b, B)$ 1 entries. Since this matrix has more than $ex(a, b, A)$ ones, A appears as a submatrix. Consider one such matrix and delete the 1 in the bottom right corner. Again, there are more than $ex(a, b, A)$ ones, so take another submatrix A and delete the 1 in the bottom right corner. Repeat this process until $ex(a, b, B) + 1$ ones have been deleted. Now consider only the ones

we deleted. There are more than $ex(a, b, B)$ of them, so at least one submatrix B must be formed. Consider one such B and take the top left corner, which is a 1 entry. This corner is also the bottom right corner of an A submatrix, so adjoining this A matrix and B matrix gives us a C matrix, and hence our original $a \times b$ 0-1 matrix that has at least $ex(a, b, A) + ex(a, b, B)$ 1 entries contains a C submatrix. ■

Lemma 2 (Tardos; [16]): Let $A = \begin{bmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & \\ & 1 \end{bmatrix}$. Then, $ex(a, b, A) \leq (b/2 + a/2) \log_2(a + b) + 2a$ and $ex(a, b, B) \leq (a/2 + b/2) \log_2(a + b) + 2b$.

Proof: We only prove the first inequality; the second follows from symmetry. We use the convention $\log x = \log_2 x$. Consider a matrix $X = \{x_{ij}\}$, let $w(X)$ be the number of 1 entries in X , and let $f(i)$ be the integer such that $x_{ij} = 0$ when $0 < j < f(i)$ and $x_{if(i)} = 1$. Let $p(i, j)$ be the largest integer less than j such that $x_{ip(i, j)} = 1$. If there does not exist such integers, these functions are undefined. Let S be the set of pairs (i, j) such that $p(i, j) > f(i)$ and $x_{ij} = 1$. Observe that $|S| + 2a \geq w(X)$. Define

$$w_1(i, j) = \log \left(\frac{j - f(i)}{p(i, j) - f(i)} \right); \quad w_2(i, j) = \log \left(\frac{j - f(i)}{j - p(i, j)} \right)$$

Consider a row r with at least two ones. Fixing r and summing w_1 over all j such that $(r, j) \in S$, we attain $\sum w_1(r, j) = \log(q - f(i)) - \log(p - f(i)) \leq \log b$, where p is the column of the second 1 entry in the r th row and q is the column of the last 1 entry in the r th row. Thus, $\sum_{(i, j) \in S} w_1(i, j) \leq a \log b$. Let i_c be the first 1 entry in column c . For the 1 entry in column c in the p th row, let $g(p)$ be the next entry in column c . Then, $p(g(i), j) \leq f(i)$, or else the submatrix A appears in X . Therefore, fixing the column and summing over all columns gives that

$$\begin{aligned} \sum_{(i, j) \in S} w_2(i, j) &= \sum_{c=1}^b \sum_{(i, c) \in S} w_2(i, c) \leq \sum_{c=1}^b \sum_{(i, c) \in S} \log \left(\frac{c - p(g(i), c)}{c - p(i, c)} \right) \\ &\leq \sum_{c=1}^b \log(c - p(t, c)) \leq \sum_{c=1}^b \log a \leq b \log a \end{aligned}$$

where t is the final 1 entry in column c . Therefore,

$$\sum_{(i, j) \in S} (w_1(i, j) + w_2(i, j)) \geq a \log b + b \log a \geq (a + b) \log(a + b)$$

Also, $w_1(i, j) + w_2(i, j) = -\log(z - z^2) \geq 2$ where $z = (j - p(i, j))/(j - f(i))$, so

$$w(X) \leq 2a + |S| \leq \frac{1}{2} \left(\sum_{(i, j) \in S} (w_1(i, j) + w_2(i, j)) \right) + 2a \leq 2a + \left(\frac{a + b}{2} \right) \log(a + b)$$

which proves the lemma. ■

Theorem 1: Any convex n -gon has at most $n \log_2 n + 4n$ unit-distanced edges.

Proof: Consider the 0 – 1 matrix corresponding to the convex n -gon. Suppose that it has

a rows and b columns. Then, $a + b = n$. Proposition 3 forbids the submatrix $\begin{bmatrix} 1 & 1 & & \\ 1 & & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}$,

and thus forbids the submatrix $\begin{bmatrix} 1 & 1 & & \\ 1 & & & \\ & 1 & & 1 \\ & & 1 & 1 \end{bmatrix}$ as well. This matrix is composed of a $\begin{bmatrix} 1 & 1 \\ 1 & \\ & 1 \end{bmatrix}$

in the top left and a $\begin{bmatrix} 1 & & 1 \\ & 1 & 1 \end{bmatrix}$ in the bottom right that only share the 1 entry in the middle.

Combining Lemmas 1 and 2, we notice that our 0 – 1 matrix has at most

$$(a + b) \log_2(a + b) + 2(a + b) = n \log_2 n + 2n$$

1 entries, meaning that there are at most $n \log_2 n + 2n$ unit distances among the set of edges connecting one vertex from one convex chain and one vertex from the other. By proposition 1, there are at most $2n$ other edges of unit length, so there are at most a total of $n \log_2 n + 4n$ unit distances in a convex n -gon. ■

4 Forbidding rectilinear polygons in 0 – 1 matrices

Theorem 2: Non self intersecting polygon matrices are forbidden.

Proof: Consider the $(1, +, -)$ matrix corresponding to a 0 – 1 non self intersecting polygon matrix and draw edges between the 1 entries to form a rectilinear polygon.

We prove a stronger statement. Start with a rectilinear polygon formed by edges in a $(1, +, -)$ matrix, but now the endpoints do not have to be 1 entries. For every vertical edge of the polygon such that the interior of the polygon lies to the right, the top endpoint of the edge is either $+$ or 1, and the bottom endpoint is $-$ or 1. Similarly, for every vertical edge such that the interior of the polygon lies to the left, the top endpoint is $-$ or 1, and the bottom endpoint is $+$ or 1. Call such a matrix a *special rectilinear matrix* and say that the *area* of this matrix is the area contained in the corresponding polygon when the vertices of the polygon are lattice points in the Cartesian coordinate plane. We prove the result by using the diagonal property and by induction on the area of the polygon.

The base case, a 2×2 matrix, which is the only matrix that can have unit area, violates the diagonal property and is therefore forbidden. Now suppose that every special rectilinear matrix with area $a - 1$ is forbidden and consider a special rectilinear matrix with area a . We can assume that the rectilinear polygon has more than four sides, or else the problem reverts to the base case, so there is an interior angle formed by three entries (which we will treat as vertices of the polygon) that is $3\pi/2$. Consider the entries a_{ik}, a_{ij}, a_{lj} with $\angle a_{ik}a_{ij}a_{lj} = 3\pi/2$. Without loss of generality, suppose $l > i$ and $k > j$. Let the vertical ray from a_{lj} through a_{ij} intersect the polygon again first at a_{hj} , with $h < i$. This divides the original polygon into

two smaller polygons, one to the left of this ray and one to the right. Let the polygon to the left be \mathcal{L} and let the polygon to the right be \mathcal{R} . If a_{hj} is a $+$ or a 1 , then \mathcal{R} is a special rectilinear polygon and is therefore forbidden by the inductive hypothesis. Similarly, if a_{hj} is a $-$, then \mathcal{L} is forbidden by the inductive hypothesis. Either way, there is a submatrix of the original polygon that is forbidden, meaning that the original polygon is forbidden, thereby completing the induction. ■

5 Forbidden cycles with a nonintersecting edge

Theorem 3: Any cycle that has a unit edge that does not intersect any other unit edge in this cycle is forbidden.

Proof: Suppose that such a cycle exists. Furthermore, suppose that this cycle does not have any subcycles; otherwise, we can either use a subcycle instead of the original cycle, or we can delete the subcycle. Convert the cycle into an $n \times n$ $(1, +, -)$ matrix. Either the top left or the bottom right entry of this matrix is a 1 . Without loss of generality, suppose the top left entry is a 1 . Now, note that each row and each column has at least two ones. With this in mind, we show that this matrix violates the obtuse angle property. Let the entry in the i th row and j th column be a_{ij} . Consider the leftmost entry in the first row other than a_{11} that is less than or equal to 1 . Call this entry a_{1w} . Now consider the set S of pairs (a_{ij}, a_{ik}) so that $a_{ij} = a_{ik} = 1$, $j < w$, and $w \leq k \leq n$. We first prove that this set is nonempty. Suppose that it is indeed empty. Draw edges connecting 1 entries in the same row or column. Consider the rectangle R with corners a_{1w} , a_{1n} , a_{nn} , and a_{nw} . Our assumption implies that any edge not in the topmost row that contains an element as an endpoint in (including the boundary of) R lies completely in R . However, this entails that there is at most one path from a_{11} to any 1 entry in R , contradicting the fact that our matrix is a cycle. Therefore, S is nonempty.

Now, consider the pair from set S , (a_{ij}, a_{ik}) , such that j is minimal. Let this pair be (a_{xy}, a_{xz}) . Then, $a_{1y} \geq 1$ by the definition of w . Each row and each column has at least two one entries, so begin drawing arrows as follows. Start from a_{xy} and point to a_{xz} . Then draw another from a_{xz} pointing to the closest 1 entry in a_{xz} 's column. Again, draw another arrow pointing from this new 1 entry to the closest 1 in this 1 entry's row, and continue this process by alternating between drawing arrows in rows and columns. Since our cycle has no subcycle, an arrow must eventually point to a_{11} , meaning an arrow must point to a_{m1} , where m is defined to be an integer such that $m > 1$ and $a_{m1} = 1$. Since a_{m1} is to the left of a_{xz} , an arrow must eventually point to the left. Consider the first such left arrow, leading from $a_{p_1q_1}$ to $a_{p_1q_2}$. Now consider the following two cases, depending on whether there is an arrow leading upwards before this left arrow or not.

Case 1: There is no up arrow before the arrow from $a_{p_1q_1}$ to $a_{p_1q_2}$. Suppose that the other 1 -entry in the q_1 th column is $a_{p_0q_1}$. Since all arrows point either right or down before that from $a_{p_1q_1}$, $p_1, p_0 \geq x$ and $q_1 \geq z \geq w$. Now if $q_2 < y$, then the pair $(a_{p_1q_2}, a_{p_1q_1})$ is in S , and thus contradicts the minimality of y . Thus, $q_2 < y$, which implies that the sextuple

$(a_{1y}, a_{1w}, a_{xy}, a_{p_1q_2}, a_{p_1q_1}, a_{p_0q_1})$ violates the obtuse angle property.

Case 2: There is an up arrow before the first left arrow. Consider the first upward arrow, and let it lead from $a_{t_1s_1}$ to $a_{t_2s_1}$. Let the other 1 entry in the t_1 th row be $a_{t_1s_0}$. An arrow leads from $a_{t_1s_0}$ to $a_{t_1s_1}$, meaning that it must be pointing to the right. All arrows point from the right before that from $a_{t_2s_1}$ and all arrows point down before that from $a_{t_1s_1}$. Hence, $t_0, t_1 \geq x$, $t_2 \geq 1$, and $s_1 \geq s_0 \geq z$. Therefore, the sextuple $(a_{1y}, a_{1w}, a_{xy}, a_{t_1s_0}, a_{t_1s_1}, a_{t_2s_1})$ violates the obtuse angle property, thereby completing the proof of Theorem 3. ■

Remark: There exist cycles of unit distanced edges in which these edges do not intersect each other, such as regular polygons with each side of unit length.

6 Exhibiting a distance-like matrix

Theorem 4: There exists a $2^m \times 2^m$ distance-like matrix that satisfies the diagonal property and the obtuse angle property with $2^{m-1}(m+1)$ ones.

PROOF: We first form the *skeleton matrix*, $[A_i]$, which is the 0 – 1 submatrix of the the

distance-like matrix. Let the matrix $[A_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and let $[A_{m+1}] = \begin{bmatrix} & & & 1 & \\ & & & 1 & [A_m] \\ & \cdots & & & \\ 1 & & & & \\ & & [A_m] & & \end{bmatrix}$.

For instance,

$$[A_2] = \begin{bmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}; \quad [A_3] = \begin{bmatrix} & & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & 1 & & 1 & \\ 1 & 1 & & 1 & \\ & 1 & 1 & & \\ 1 & & & & \end{bmatrix}$$

$[A_4], [A_5], \dots$ are created recursively. Let $|[A_k]|$ be the number of 1 entries in the matrix $[A_k]$. Then, $|[A_{k+1}]| = 2|[A_k]| + 2^k$, and hence, by induction, $|[A_m]| = 2^{m-1}(m+1)$. Füredi and Hajnal used almost the same matrix to give an example of a matrix that does not have $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ as a submatrix in [9], but instead let $[A_1] = \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix}$.

Now we begin filling in the zero cells to form the foundation of the distance-like matrix. In the *simplified matrix*, we put a m -tuple of integers in each of the entries of $[A_m]$. The entry in the p th row and q th column is the m -tuple $(a_{pq,1}, a_{pq,2}, a_{pq,3}, \dots, a_{pq,m})$, which will be chosen more carefully later. Next, we pick a m -tuple of reals, $(x_1, x_2, x_3, \dots, x_m)$, where these

entries will also be chosen more carefully later. The entry in the i th row and j th column of our distance-like matrix will be $1 + \sum_{k=1}^m a_{ij,k} x_k$, and will show that we can put m -tuples in A_k for all $1 \leq k \leq m$ so that the obtuse angle and diagonal properties. Define the (s, t) *simplified matrix* with $s \leq t$ to be the distance-like matrix corresponding to the skeleton $[A_t]$ with only the coefficient of x_s (i.e., the matrix which has $a_{pq,s}$ in the p th row and q th column).

First we create the (r, r) simplified matrix for each r . Call this matrix $[S_r]$. Let $[Y_{r-1}]$ be a $2^{r-1} \times 2^{r-1}$ matrix so that the entry in the i th row and j th column is y_{ij} . Let $y_{ij} = 0$ when $i + j = 2^{r-1} + 1$. When $i + j = 2^{r-1} + l + 1$ with $l > 0$, let $y_{ij} = i + 2^{2^{i-1}-j} l z$, where $z = \frac{1}{5^{5^r}}$. If $i + j \leq 2^{r-1}$, then set $y_{ij} = -2^{5^r - 2i - 2j}$. Observe that, below the diagonal of zeroes, the entries of $[X_k]$ are increasing from top to bottom (this holds because y_{ij} is between i and $i + 1$ for each y_{ij} below the diagonal) and from left to right. The same holds for entries above the diagonal of zeroes, which are increasing from left to right and top to bottom in all of $[X_{r-1}]$. Let $[Z_{r-1}]$ be another $2^{r-1} \times 2^{r-1}$ matrix with the entry in the i th row and the j th column being equal to $-5^{5^r} i j$. Observe that the entries in $[Y_{r-1}]$ are decreasing top to bottom and left to right. Now let $[S_r] = \begin{bmatrix} Y_{r-1} & 0 \\ 0 & Z_{r-1} \end{bmatrix}$. For any a and b , let $S_{a,b}$ be the (b, a) simplified matrix when $a < b$. Let $S_{a,b} = \begin{bmatrix} 0 & S_{a,b-1} \\ S_{a,b-1} & 0 \end{bmatrix}$. This way, we can form the distance-like matrix by combining all of the simplified matrices; we refer to this matrix as $[D'_m]$. Now we define the x -series recursively. Suppose that we have an $(m-1)$ -tuple, $(x_1, x_2, \dots, x_{m-1})$ corresponding to the skeleton matrix $[A_{m-1}]$ that satisfies the obtuse angle and diagonal property. We show that there exists an m -tuple $(x'_1, x'_2, \dots, x'_{m-1}, x_m)$ that corresponds to $[A_m]$ and satisfies both properties. Notice that, for any l , if we divide any x_l by some number, then both the obtuse angle property and also the diagonal property still hold. Let $x_m = 10^{-10^{10^m}}$, which is sufficiently small so as to ensure that all entries in $[D'_m]$ are positive. Divide all of the x_i , with $1 \leq i \leq m-1$, by a sufficiently large number, such as $1/x_m^2$ so as to make all of the other entries approximately equal to 1 when compared to the entries in $[Y_{m-1}]$ and $[Z_{m-1}]$. Now, subtract 1 from each entry in this distance-like matrix and call the new matrix $[D_m]$. Doing this neither disrupts the obtuse angle property nor the diagonal property.

Proceed by induction on the size of the matrix to prove that this matrix satisfies both properties. It is clear for the base case of $m = 0$. Suppose that this matrix works for all $1 \leq i \leq n$. Now consider $[D_{n+1}]$. Divide $[D_{n+1}]$ into four disjoint $2^n \times 2^n$ submatrices. Let the top left submatrix $[S_n]$, the bottom left submatrix $[V_n]$, the bottom right submatrix $[U_n]$, and the top right submatrix $[T_n]$. We can assume that the entries in $[V_n]$ and $[T_n]$ are approximately 0 in comparison to the entries of $[U_n]$ and $[S_n]$.

First consider the diagonal property. Suppose there are four entries, $a_{ij}, a_{ik}, a_{lj}, a_{lk}$ so that $a_{ij} + a_{lk} \geq a_{ik} + a_{lj}$ with $i < l$ and $j < k$. Suppose first that a_{lk} is in $[U_n]$. If a_{ij} is in $[V_n]$, then a_{lj} is in $[V_n]$ and a_{ik} is in $[U_n]$. Since the entries in $[U_n]$ are decreasing from top to

bottom, $a_{ik} - a_{lk} > 0$. Since a_{lj} and a_{ij} are approximately 0 in comparison to this difference, $a_{ik} - a_{lk} > a_{ij} - a_{lj}$, and thus the diagonal property is satisfied. Similarly, using the fact that entries are decreasing from left to right in $[U_n]$, we can show that a_{ij} cannot be in $[T_n]$. If a_{ij} is in $[U_n]$, then a_{ik} and a_{lj} are also in $[U_n]$, making the diagonal property easy to verify using the closed form of the entries in $[U_n]$. Now, if a_{ij} is in $[S_n]$, then a_{ik} is in $[T_n]$ and a_{lj} is in $[V_n]$. Any entry in $[U_n]$ is far greater in magnitude than any entry in $[S_n]$, but is also negative, so $a_{ij} + a_{lk} < 0$, and $a_{ik} + a_{lj}$ is approximately 0, so the inequality is still satisfied. Hence, assume a_{lk} is not in $[U_n]$. Next, suppose that a_{lk} is in $[V_n]$. Then, if a_{ij} is in $[V_n]$, then a_{ik} and a_{lj} are also in $[V_n]$, so the inductive hypothesis tells us that the diagonal property is satisfied. If a_{ij} is in $[S_n]$, then a_{ik} is in $[S_n]$ and a_{lj} is in $[V_n]$. Since entries in $[S_n]$ are increasing from left to right and entries in $[V_n]$ are, in comparison, approximately zero, the diagonal property is again satisfied. Therefore, a_{lk} is not in $[V_n]$. Similarly, a_{lk} cannot be in $[T_n]$, implying that it is in $[S_n]$, which implies that a_{ij} , a_{ik} , and a_{lj} are also in $[S_n]$. Using the closed formulae for the entries in $[S_n]$, it is easy to check that the diagonal property still holds. Thus, there is no quadruple $(a_{ij}, a_{ik}, a_{lj}, a_{lk})$ that violates the diagonal property.

Next, we prove that this matrix satisfies the obtuse angle property by using contradiction. The entries in $[U_n]$ are decreasing from top to bottom and from left to right. Moreover, any entry in $[U_n]$ is less than any entry not in $[U_n]$, meaning that the bottom right corner of any acute angle matrix cannot be in $[U_n]$. If the corner is in $[S_n]$, then all entries of the acute angle matrix are in $[S_n]$. However, entries in S_n increase from left to right, meaning that the top corner of this acute angle matrix is strictly less than the entry to the right of it in the acute angle matrix, which is a contradiction. Therefore the bottom right corner of the acute angle matrix cannot reside in $[S_n]$. Thus, the bottom right corner is either in $[T_n]$ or in $[V_n]$. Since both proofs are the same, without loss of generality, suppose that this corner is in $[V_n]$. If the top left corner of this acute angle matrix were in $[V_n]$, the inductive hypothesis would yield a contradiction. Therefore, the top left corner of the acute angle matrix is in $[S_n]$. However, any entry in $[S_n]$ to the right of this corner has a value greater than this corner, implying that the acute angle submatrix cannot exist, which is again a contradiction. Hence, the obtuse angle property is not violated either. ■

7 Discussion

Below we provide some general discussion regarding our results, including open problems and possible reasons as to why we could not progress further.

7.1 Discussion on Füredi's forbidden matrices

First, we mention a corollary of Theorems 1 and 2.

Corollary 2: The only allowable 3×3 $0-1$ submatrix with at least 6 ones is $\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{bmatrix}$.

PROOF: Suppose a row has three ones. Since there are at least three ones remaining, one other row contains at least two ones. However, this has the submatrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which is forbidden by the diagonal condition. Therefore, each row and each column has at most two 1 entries. Since there are a total of at least six 1 entries, each row and each column have exactly two 1 entries. Since only 2×2 cycle is forbidden, this matrix forms a 3×3 cycle. By Theorem 1, this cycle can have neither the top left nor bottom right entry as a 1, meaning that the second and last entries in the first row and the first two entries of the last row are 1s. This means that the middle column has two 1s, so it cannot accomodate any more 1s. Thus, the middle row must have 1s in its first and last entries, yielding $\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{bmatrix}$, which actually corresponds 0 – 1 matrix of a convex hexagon.

In [10], Füredi forbade the matrix $\begin{bmatrix} 1 & 1 & \\ 1 & & 1 \end{bmatrix}$ under the condition that there is an acute angle. Then, he broke the plane into strips to assure that such an acute angle exists and used a slightly weaker form of Lemma 2 in Theorem 3 to attain an upper bound of $2\pi n \log_2 n - \pi n$ unit-distanced edges. It is interesting to note that this forbidden matrix is contained in the two matrices $\begin{bmatrix} 1 & 1 & \\ 1 & & 1 \\ & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} & 1 & 1 \\ 1 & 1 & \\ 1 & & 1 \end{bmatrix}$, both of which are forbidden by corollary 2. However, the matrices $\begin{bmatrix} 1 & & 1 \\ 1 & 1 & \end{bmatrix}$ and $\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \end{bmatrix}$ are not forbidden by Füredi and are contained in the allowable matrix $\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{bmatrix}$. If we imposed the acute angle condition Füredi imposed, we would still be able to draw a pentagon corresponding to each of these matrices. Therefore, it seems that while Füredi’s ”forbidden” matrix is not forbidden under all conditions, it is a reasonably good estimator of the types of 3×3 matrices that are forbidden.

7.2 Reasons as to why we cannot progress further

Among the reasons as to why we cannot progress further is that matrices similar to the distance-like matrix exhibited in Theorem 4 have entries very close together (within 10^{-10^n} units apart), meaning that the corresponding quadrilaterals become very difficult to draw. Moreover, we have not been able to find any submatrices in any of the skeleton matrices mentioned above that are forbidden. For instance, the *intertwining cycles*

$$\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{bmatrix}; \quad \begin{bmatrix} & & 1 & 1 \\ & 1 & & 1 \\ 1 & & 1 & \\ 1 & 1 & & \end{bmatrix}$$

appear constantly in our skeleton matrices and can be drawn into corresponding convex chains when the matrices have small dimensions. However, we have not been able to show that all intertwining matrices can correspond to convex polygons. Note also that the skeleton $[A_1]$ has many possibilities, such as $\begin{bmatrix} 1 & & \\ & 1 & 1 \\ & 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{bmatrix}$, thereby making the $[A_i]$ different as well.

If one is to prove that the number of unit distances in a convex n -gon is indeed $\Theta(n)$, then one may need to show that many intertwining matrices cannot exist simultaneously. For instance, it might be helpful to know whether there exists a polygon such that each vertex of the polygon is a vertex of at least k intertwining cycles, where k is a large integer. Also, if the number of unit distanced edges is linear, then the proof would require different techniques than those exhibited in this paper, because Theorem 4 exhibits a distance-like matrix that satisfies all of the conditions we set in Section 2.

7.3 Open Questions

Below we provide additional questions that are, as far as we know, open.

- (a) Is there a forbidden 0–1 matrix that is the skeleton of a distance-like matrix that satisfies both the diagonal and obtuse angle property?
- (b) In [1], Ábrego and Fernández-Merchant proved that the number of unit-distanced edges in centrally symmetric polygons is at most $2n - 3$. What is the maximum number of unit-distanced edges in a convex polygon that is symmetric with respect to a line?

Remark: The latter question was motivated by the fact that the skeleton matrices described in Theorem 4 and the intertwining cycles are the same as their transposes. Hence, it may be possible to convert them into polygons that are symmetric with respect to a line. The 3×3 intertwining cycle, for instance, can be drawn into a hexagon that is symmetric with respect to a line.

References

- [1] B. M. Ábrego and S. Fernández-Merchant. The unit distance problem for centrally symmetric convex polygons. *Discrete and Computational Geometry*, **28** (2002), no. 4, 467-473.
- [2] E. Altman, On a problem of P. Erdős, *Amer. Math. Monthly*, **70** (1963), 148-154.
- [3] P. Brass, G. Karolyi, and P. Valtr, A Turán-type extremal theory of convex geometric graphs, *Discrete and Computational Geometry The Goodman-Pollack Festschrift*, (2003), 275-300.

- [4] P. Brass and J. Pach, The maximum number of times the same distance can occur among the vertices of a convex n -gon is $O(n \log n)$, *J. Combin. Theory Ser. A*, **94** (2001), 178-179.
- [5] H. Edelsbrunner and P. Hajnal, A lower bound on the number of unit distances between points of a convex polygon, *J. Combin. Theory Ser. A*, **56** (1991), 312-316.
- [6] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly*, **53** (1946), 249-250.
- [7] P. Erdős and P. C. Fishburn, Multiplicities of interpoint in finite planar sets, *Discrete and Applied Mathematics* **60**, (1995), 141-147.
- [8] P. Erdős and L. Moser, Problem 11, *Canadian Math. Bulletin*, **2** (1959), 53.
- [9] Z. Füredi, The maximum number of unit distances in a convex n -gon, *J. Combin. Theory Ser. A*, **55** (1990), 316-320.
- [10] Z. Füredi and P. Hajnal, Davenport-Schinzel Theory of Matrices, *Discrete Mathematics*, **103** (1992), no. 3, 233-251.
- [11] B. Keszegh, Forbidden submatrices in 0 – –1 matrices, *Master's thesis*, Eotvos Lorand University, (2005).
- [12] B. Keszegh, On linear forbidden submatrices, *Journal of Combinatorial Theory Ser. A*, **116** (2009), 232-241.
- [13] L. Moser, On different distances determined by n points, *American Mathematical Monthly*, **59** (1952), 85-91.
- [14] J. Pach and G. Tardos, Forbidden paths and cycles in ordered graphs and matrices, *Israel J. Math.*, **155** (2006), 309-334.
- [15] Szemerédi and W.T. Trotter Jr., Extremal Problems in Discrete Geometry, *Combinatorica*, **3** (1983), 381-397.
- [16] G. Tardos, On 0-1 matrices and small excluded submatrices, *Journal of Combinatorial Theory Ser. A*, **111** (2005), no. 2, 266-288.
- [17] G. Toussaint, Computational geometric aspects of rhythm, melody, and voice-leading, *Computational Geometry: Theory and Applications*, **43** (2010), 2-22.